Lecture 15

When Intuition Differs from Relative Frequency

Revisiting Relative Frequency

- The relative frequency interpretation of probability provides a precise answer to certain probability questions.

- As long as we understand the physical assumptions underlying an uncertain process, we can also agree on the probabilities of various outcomes.

Thought Question 1:

Do you think it ***likely*** that anyone will ever ***win a state lottery twice*** in a lifetime?

Thought Question 2:

How many people do you think would need to be in a group in order to be at least 50% certain that two of them will have the **same birthday**?

Thought Question 3:

You test positive for rare disease, your original chances of having disease are 1 in 1000.

The test has a 10% false positive rate and a 10% false negative rate => ***whether you have disease or not, test is 90% likely to give a correct answer***.

Given you tested positive, what do you think is the ***probability that you actually have disease***? Higher or lower than 50%?

Thought Question 4:

If you were to flip a fair coin six times, which sequence do you think would be most likely:

HHHHHH or HHTHTH or HHHTTT?

Thought Question 5:

Which one would you choose in each set? (Choose either A or B and either C or D.) Explain your answer.

A. A gift of $240, guaranteed
B. A 25% chance to win $1000 and a 75% chance of getting nothing

C. A sure loss of $740
D. A 75% chance to lose $1000 and a 25% chance to lose nothing

18.1 Revisiting Relative Frequency

- ***Relative-frequency interpretation*** provides a precise answer to certain probability questions.
- Often the physical situation lends itself to computing a relative-frequency probability, but people ignore that information.

18.2 Coincidences

Are Coincidences Improbable?

A coincidence is a surprising concurrence of events, perceived as meaningfully related, with no apparent causal connection. (*Source:* Diaconis and Mosteller, 1989, p. 853)

Example 3: Winning the Lottery Twice

- NYT story of February 14, 1986, about Evelyn Marie Adams, who won the NJ lottery twice in short time period.
- NYT claimed that the odds of one person winning the top prize twice were about 1 in 17 trillion.

Source: Moore (1991, p. 278)

Someone, Somewhere, Someday

*What is **not improbable** is that someone, somewhere, someday will experience those events or something similar.*

We often ask the wrong question …

- The ***1 in 17 trillion*** is the probability that a ***specific*** individual who plays the NJ state lottery exactly twice will win both times (Diaconis and Mosteller, 1989,p. 859).
- ***Millions of people play lottery*** every day, so not surprising that someone, somewhere, someday would win twice.
- Stephen Samuels and George McCabe calculated … ***at least a 1 in 30 chance*** of a double winner in a 4-month period and better than even odds that there would be a double winner in a 7-year period somewhere in the U.S.

Lincoln & Kennedy

- Abraham Lincoln was elected to Congress in 1846.
- John F Kennedy was elected to Congress in 1946.
- Abraham Lincoln was elected President in 1860.
- John F. Kennedy was elected President in 1960.
- The names Lincoln and Kennedy each contain seven letters.
- Both were particularly concerned with civil rights.

Lincoln & Kennedy

- Both wives lost a child while living in the White House.
- Both Presidents were shot on a Friday.
- Both Presidents were shot in the head.
- Lincoln's secretary was named Kennedy.
- Kennedy's secretary was named Lincoln.
- Both were assassinated by Southerners.

Lincoln & Kennedy

- Both were succeeded by Southerners named Johnson.
- Andrew Johnson, who succeeded Lincoln, was born in 1808.
- Lyndon Johnson, who succeeded Kennedy, was born in 1908.
- John Wilkes Booth, who assassinated Lincoln, was born in 1839.
- Lee Harvey Oswald, who assassinated Kennedy, was born in 1939.

Lincoln & Kennedy

- Both assassins were known by their three names.
- Both names are composed of fifteen letters.
- Lincoln was shot at the theatre named 'Ford.'
- Kennedy was shot in a car called 'Lincoln.'
- Booth ran from the theatre and was caught in a warehouse.
- Oswald ran from a warehouse and was caught in a theatre.
- Booth and Oswald were assassinated before their trials.

Lincoln & Kennedy

- And here's the clincher.
- A week before Lincoln was shot, he was in Monroe, Maryland.
- A week before Kennedy was shot, he was with Marilyn Monroe.
- Oh…and on the day he died Lincoln pardoned a man named…
- Patrick Murphy

Example 4: Sharing the Same Birthday

How many people need to be in a group to be at least 50% sure that two share the same birthday?

Answer: 23 people.

Most answer much higher because thinking about the probability that someone will have ***their*** birthday.

Probability the first three people have different birthdays: probability the second person does not share a birthday with the first (364/365) ***and*** third person does not share a birthday with either of first two (363/365) => $(364)(363)/(365)^2$ =.9918

Probability that none of the 23 people share a birthday: $(364)(363)(362) \cdots (343)/(365)^{22} = 0.493$.

Probability at least 2 people share a birthday: 1–.493 =.507.

The Birthday Problem

What is the probability that (at least) two people in our class share a birthday?

Notice that the only way to avoid two people having the same birthday is if all 42 people in our class have different birthdays.

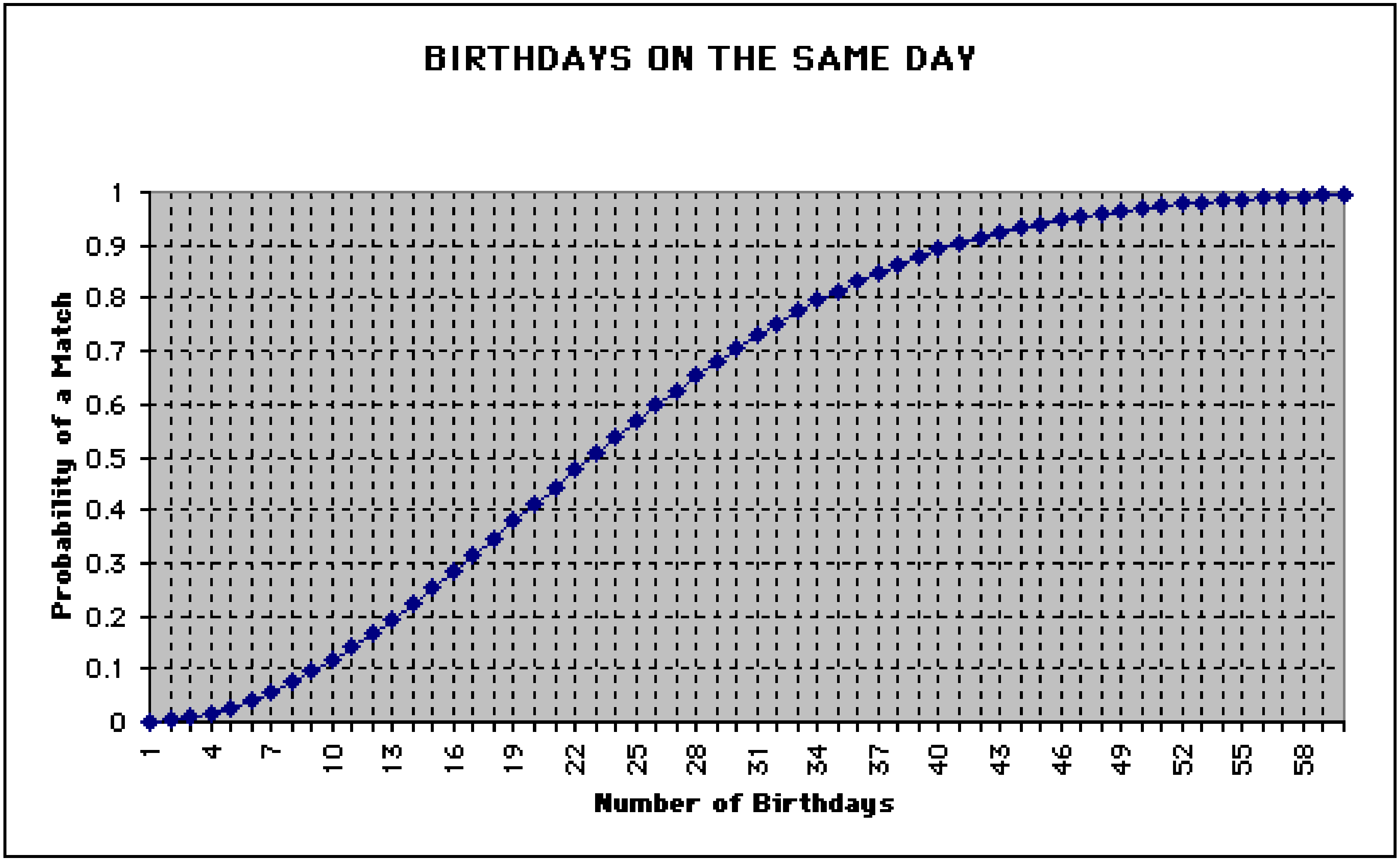
Consider only 3 people. The probability that these three people have different birthdays is the probability that the second person does not share a birthday with the first, which is 364/365, and the third person does not share a birthday with either of the first two, which is 363/365.

Using Independence (one person's birthday is clearly independent of another persons birthday) and continuing this logic….

The probability that none of the 42 people in our class share a birthday is

$(364)(363)(362)(361)\bullet \ldots \bullet(324)/(365)^{41}=0.086$

So, the probability that at least two people in our class share a birthday is 1-0.086=0.914.



BIRTHDAYS ON THE SAME DAY
Probability of a Match
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1
4
7
10
13
16
19
22
25
28
31
34
37
40
43
46
49
52
55
58
Number of Birthdays

Most Coincidences Only Seem Improbable

- Coincidences seem improbable only if we ask the probability of that ***specific event occurring at that time to us***.
- If we ask the probability of it occurring some time, to someone, the probability can become quite large.
- Multitude of experiences we have each day => not surprising that ***some*** may appear ***improbable***.

17.3 The Gambler's Fallacy

People think the long-run frequency of an event should apply even in the short run.

Tversky and Kahneman (1982) define a related idea - the belief in the law of small numbers, *"according to which [people believe that] even small samples are highly representative of the populations from which they are drawn."* (p. 7) …

"in considering tosses of a coin for heads and tails … people regard the sequence HTHTTH to be more likely than the sequence HHHTTT, which does not appear to be random, and also more likely than HHHHTH, which does not represent the fairness of the coin" (p. 7)

The Gambler's Fallacy

Independent Chance Events Have No Memory

Example:

People tend to believe that a string of good luck will follow a string of bad luck in a casino. However, making ten bad gambles in a row doesn't change the probability that the next gamble will also be bad.

The Gambler’s Fallacy

When It May Not Apply

The Gambler’s fallacy applies to independent events. It may not apply to situations where knowledge of one outcome affects probabilities of the next.

Example:
In card games using a single deck, knowledge of what cards have already been played provides information about what cards are likely to be played next.

18.4 Confusion of the Inverse

Malignant or Benign?

- Patient has lump, physician assigns 1% chance it is malignant.
- Mammograms are 80% accurate for malignant lumps and 90% accurate for benign lumps.
- Mammogram indicates lump is malignant.

What are the chances it is truly malignant?

In study, most physicians said about 75%, but it is **only 7.5%!**

Confusion of the Inverse: Physicians were confusing the probability of *cancer given a positive mammogram* result with its inverse, the probability of a *positive mammogram given the patient has cancer*.

Source: Plous (1993, p. 132)

Confusion of the Inverse

Determining the Actual Probability

Hypothetical Table for 100,000 women with lump and prior probability of it being malignant is 1%.

	Test Shows Malignant	Test Shows Benign	Total
Actually Malignant	800	200	1,000
Actually Benign	9,900	89,100	99,000
Total	10,700	89,300	100,000

Given test shows malignant, probability of actually malignant is: 800/10,700 = 0.075.

Confusion of the Inverse

The Probability of False Positives

If *base rate* for disease is very low and test for disease is less than perfect, there will be a relatively high probability that a positive test result is a ***false positive***.

To determine probability of a positive test result being accurate, you need:

1. **Base rate** or probability that you are likely to have disease, without any knowledge of your test results.
2. **Sensitivity** of the test – the proportion of people who correctly test positive when they actually have the disease
3. **Specificity** of the test – the proportion of people who correctly test negative when they don't have the disease

Use hypothetical table or Bayes' Rule Formula.

Case Study 18.1: Streak Shooting in Basketball: Reality or Illusion?

Study:

Generated phony sequences of 21 alleged 'hits and misses' in shooting baskets. Showed to 100 BB fans. Asked to classify ***chance shooting, streak shooting,*** or ***alternating shooting***.

65% thought a chance shooting sequence was in fact ***streak shooting***.

Try It:

Which 21 shot sequence is more likely to be result of ***chance shooting***?

Sequence 1: FFSSSFSFFFSSSSFSFFFSF

Sequence 2: FSFFSFSFFFSFSSFSFSSSF

In 1: Of 20 throws that have a preceding throw, exactly 10 are different.

In 2: 14 of 20 or 70% of shots differ from previous shot.

Answer: Sequence 1.

Source: Tversky and Gilovich, Winter 1989.

Case Study 18.1: Streak Shooting in Basketball: Reality or Illusion?

When shooting free throws, does a player have a better chance of making his second shot after making his first shot than after missing his first shot? (1989, p. 20)

68% said YES

Tversky and Gilovich gathered and analyzed in variety of ways:

Our research ***does not tell us anything in general about sports****, but it does suggest a* ***generalization about people****, namely that they* ***tend to "detect" patterns even where none exist,*** *and to overestimate the degree of clustering in sports events, as in other sequential data. We attribute the discrepancy between the observed basketball statistics and the intuitions of highly interested and informed observers to a* ***general misconception of the laws of chance*** *that induces the expectation that random sequences will be far more balanced than they generally are, and creates the illusion that there are patterns of streaks in independent sequences. (1989, p. 21)*

(Mis)perceptions of randomness

- Distributions of heads and tails:
  - HTTTHTTTHTTHHHTHHTTHH
  - HTHTHHTTHTHTHTTHTHTHH
- Representativeness leads to misidentification of randomness
- Implications...

Hot hand phenomenon

“If I’m on, I find that confidence just builds... you feel nobody can stop you. it’s important to hit that first one, especially if it’s a swish. Then you hit another, and...you feel like you can do anything.”

~Lloyd Free (a.k.a. World B. Free)

What is the “Hot hand”?

- Belief that success breeds success, Failure breed failure
- 100 basketball fans:
  - 91% thought “player has better chance of making a shot after having just made his last two or three shots than he does after having just missed his last two or three shots.”
  - Given a player who makes 50% of his shots, subjects thought that shooting percentage would be...
    - 61% after having just made a shot
    - 42% after having just missed a shot
  - 84% thought “it’s important to pass the ball to someone who has just made several shots in a row.”

(Gilovich, Vallone, & Tversky, 1985)

Does the "Hot hand" exist?

•Calculate probability of making a shot after missing previous 1, 2, or 3 shots and after making the previous 1, 2, or 3 shots. (Gilovich, Vallone, & Tversky, 1985)

Table 2.1 Probability of Making a Shot Conditioned on the Outcome of Previous Shots for Nine Members of the 76ers

Player	*P*(x|*ooo*)	*P*(x|*oo*)	*P*(x|*o*)	*P*(*x*)	*P*(*x*|*x*)	*P*(x|*xx*)	*P*(x|*xxx*)	*r*
C. Richardson	.50	.47	.56	.50	.49	.50	.48	−.02
J. Erving	.52	.51	.51	.52	.53	.52	.48	.02
L. Hollins	.50	.49	.46	.46	.46	.46	.32	.00
M. Cheeks	.77	.60	.60	.56	.55	.54	.59	−.04
C. Jones	.50	.48	.47	.47	.45	.43	.27	−.02
A. Toney	.52	.53	.51	.46	.43	.40	.34	−.08
B. Jones	.61	.58	.58	.54	.53	.47	.53	−.05
S. Mix	.70	.56	.52	.52	.51	.48	.36	−.02
D. Dawkins	.88	.73	.71	.62	.57	.58	.51	−.14
Mean =	.56	.53	.54	.52	.51	.50	.46	−.04

NOTE: x = a hit; o = a miss. r = the correlation between the outcomes of consecutive shots

Same goes for gambling: "The Gambler's Fallacy"

What the hot-hand results mean:

•The independence between successive shots, of course, does not mean that basketball is a game of chance rather than skill, nor should it render the game less exciting to play, watch, or analyze. It merely indicates that the probability of a hit is largely independent of the outcome of previous shots, although it surely depends on other parameters such as skill, distance to the basket, and defensive pressure. This situation is analogous to coin tossing where the outcomes of successive tosses are independent but the probability of heads depends on measurable factors such as the initial position of the coin, and its angular and vertical momentum. Neither coin tossing nor basketball are inherently random, once all the relevant parameters are specified. In the absence of this information, however, both processes may be adequately described by a simple binomial model. A major difference between the two processes is that it is harder to think of a credible mechanism that would create a correlation between successive coin tosses, but there are many factors (e.g., confidence, fatigue) that could produce positive dependence in basketball. The availability of plausible explanations may contribute to the erroneous belief that the probability of a hit is greater following a hit than following a miss. (Gilovich et al. 1985, pp. 312 - 313)

18.5 Using Expected Values To Make Wise Decisions

If you were faced with the following alternatives, **which would you choose?** Note that you can choose **either A or B and either C or D**.

A. A gift of $240, guaranteed

B. A 25% chance to win $1000 and a 75% chance of getting nothing

C. A sure loss of $740

D. A 75% chance to lose $1000 and a 25% chance to lose nothing

- **A versus B:** majority chose sure gain A. Expected value under choice B is $250, higher than sure gain of $240 in A, yet people prefer A.
- **C versus D:** majority chose gamble rather than sure loss. Expected value under D is $750, a larger expected loss than $740 in C.
- People **value sure gain**, but willing to **take risk to prevent loss**.

Source: Plous (1993, p. 132)

Using Expected Values To Make Wise Decisions

If you were faced with the following alternatives, **which would you choose?** Note that you can choose **either A or B and either C or D**.

Alternative A:	A 1 in 1000 chance of winning $5000
Alternative B:	A sure gain of $5
Alternative C:	A 1 in 1000 chance of losing $5000
Alternative D:	A sure loss of $5

- **A versus B:** 75% chose A (*gamble*). Similar to decision to buy a lottery ticket, where sure gain is keeping $ rather than buying a ticket.
- **C versus D:** 80% chose D (*sure loss)*. Similar to success of insurance industry. Dollar amounts are important: sure loss of $5 easy to absorb, while risk of losing $5000 may be equivalent to risk of bankruptcy.

Source: Plous (1993, p. 132)

Case Study 18.2: How Bad Is a Bet on the British Open?

If people made decisions solely on the basis of maximizing expected dollar amounts, they would not bet.

Study:

- Look at odds given for 1990 British Open golf tournament "by one of the largest betting shops in Britain" (p. 25) to see how much shop stood to gain and bettors stood to lose.
- The **bookmaker sets odds** on each of the possible outcomes, which in this case were individual players winning the tournament.
- **If odds are 50 to 1** for Jack Nicklaus and you pay one dollar to play, two possible outcomes are that you **gain $50 or you lose $1**.

Source: Larkey, 1990, pp. 24-36.

Case Study 18.2: How Bad Is a Bet on the British Open?

Probability of winning required for break-even expected value.

	Player	Odds	Probability
1.	Nick Faldo	6 to 1	.1429
2.	Greg Norman	9 to 1	.1000
3.	Jose-Maria Olazabal	14 to 1	.0667
4.	Curtis Strange	14 to 1	.0667
5.	Ian Woosnam	14 to 1	.0667
6.	Seve Ballesteros	16 to 1	.0588
7.	Mark Calcavecchia	16 to 1	.0588
8.	Payne Stewart	16 to 1	.0588
9.	Bernhard Langer	22 to 1	.0435
10.	Paul Azinger	28 to 1	.0345
11.	Ronan Rafferty	33 to 1	.0294
12.	Fred Couples	33 to 1	.0294
13.	Mark McNulty	33 to 1	.0294

Source: Larkey, 1990, pp. 24-36.

- Odds are n to 1 for a player, someone who bets on player will have an expected gain (or loss) of zero if the probability of player winning is $1/(n + 1)$.
- **If** bookmaker set **fair odds**, then probabilities for all players should sum to 1.00.
- Sum of all probabilities (only some listed) is 1.27.
- House has definite advantage, even after "handling fee."

For Those Who Like Formulas

Conditional Probability

The *conditional probability* of event A, given knowledge that event B happened, is denoted by $P(A|B)$.

Bayes' Rule

Suppose A_1 and A_2 are complementary events with known probabilities. In other words, they are mutually exclusive and their probabilities sum to 1. For example, they might represent presence and absence of a disease in a randomly chosen individual. Suppose B is another event such that the conditional probabilities $P(B \mid A_1)$ and $P(B \mid A_2)$ are both known. For example, B might be the probability of testing positive for the disease. We do not need to know $P(B)$.

Then Bayes' Rule determines the conditional probability in the other direction:

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

For example, Bayes' Rule can be used to determine the probability of having a disease, *given* that the test is positive. The base rate, sensitivity, and specificity would all need to be known. Bayes' Rule is easily extended to more than two mutually exclusive events, as long as the probability of each one is known, and the probability of B conditional on each one is known.